Geometric realizations of homotopic paths over curved surfaces

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Abstract. This paper introduces geometric realizations of homotopic paths over simply-connected surfaces with non-zero curvature as a means of comparing and measuring paths between antipodes with either a Feynman path integral or Woodhouse contour integral, resulting in a number of extensions of the Borsuk Ulam Theorem. All realizations of homotopic paths reside on a Riemannian surface $S$, which is simply-connected and has non-zero curvature at every point in $S$. A fundamental result in this paper is that for any pair of antipodal surface points, a path can be found that begins and ends at the antipodal points. The realization of homotopic paths as arcs on a Riemannian surface leads to applications in Mathematical Physics in terms of Feynman path integrals on trajectory-of-particle curves and Woodhouse contour integrals for antipodal vectors on twistor curves. Another fundamental result in this paper is that the Feynman trajectory of a particle is a homotopic path geometrically realizable as a Lefschetz arc.

1. Introduction

This paper introduces a path-Borsuk-Ulam Theorem, stemming from three main forms of paths over curved surfaces that have been identified, namely,

1° Poincaré Contour paths were introduced by Poincaré in 1892 in his analysis situs paper [17]. In a contour path, each subpath is an infinitely small contour on a manifold [17, p. 240]. Recently, N.M.J. Woodhouse [23] introduced contour integrals defined on twistor curves on a complex manifold.

2° Whitehead Homotopic paths were introduced during the late 1940s by J.H.C. Whitehead [21, 22] and S. Lefschetz [6], elaborated in [14–16]. For Whitehead, a path is a continuous map $h : [0, 1] \rightarrow S$, i.e., a map from the unit interval to a space $S$. For Lefschetz, a homotopic path $h$ in an arcwise connected space $S$ is...
simply a map of a directed (= oriented) closed arc $v_0, \ldots, v_k$ into $S$ [6, p. 158]. A space is arcwise connected provided every vector in the space $S$ is on a path containing an initial vector and a terminal vector such as the arcs in Figure 1.

3° **Feynman paths** were introduced by R.P. Feynman in his thesis completed in 1942 [3, p. xiv]. A Feynman path is a trace of the trajectory of a particle between fixed endpoints [3, p. xiv], providing a framework for a path integral, also introduced by Feynman [3] and elaborated by R.P. Feynman and A.R. Hibbs in [4]. A Penrose path over a twistor curve (from R. Penrose’s 1968 paper [13]) and its refinement by R.S. Ward in his 1977 thesis [20] supervised by Penrose, is a form of Feynman path in which the trajectory of a particle is over a twistor curve.

The original Borsuk-Ulam Theorem (BUT) [2] from K. Borsuk in 1933 is given in terms of antipodal vectors $\vec{p}, -\vec{p}$ on the surface of an $n$-dimensional Euclidean sphere $S^n$, defined by

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}, \ x_1^2 + \cdots + x_{n+1}^2 = 1, \ n \geq 2\}.$$  

Points on the surface of a sphere are **antipodal** provided the points are diametrically opposite each other. Examples of antipodal vectors are the poles on the surface of a planet.

In 1933, K. Borsuk introduced the following theorem.

**Theorem 1.1. (Borsuk-Ulam Theorem)** [2, p. 178] For every continuous map $f : S^n \to \mathbb{R}^n$, there exists $\vec{p} \in S^n$ such that $f(\vec{p}) = f(-\vec{p})$.

**Remark 1.2.** Theorem 1.1 is a translation from German, which is given by J. Matoušek [9, p. 21].

**Remark 1.3.** The basis for Theorem 1.1 came from K. Borsuk’s thesis completed in 1930 [1]. Ulam is credited by Borsuk (in a footnote [2, p. 178]) with the idea codified in Theorem 1.1, which Ulam stated as a conjecture. In effect, Borsuk proved Ulam’s conjecture in 1933. In 1930, L. Lusternik and S. Shnirel’man introduced the nonvoid intersection of sets of closed surface curves that have antipodal vectors in common.

**Theorem 1.4. (Lusternik-Shnirel’man Theorem)** [7] For any cover $F_1, \ldots, F_{n+1}$ of the sphere $S^n$ by $n + 1$ closed sets, there is at least one set containing a pair of antipodal points common to $F_i, -F_i$ (i.e., $F_i \cap -F_i \neq \emptyset$).

**Remark 1.5.** Theorem 1.4 is a translation from Russian, which is given by J. Matoušek [9, p. 21].

Theorem 1.4 contrasts with Theorem 1.1. In the Lusternik-Shnirel’man Theorem 1.4, there is a closed set $F_i$ that is a cover of a sphere $S^n$ and that has an opposite set $-F_i$, in which the sets $F_i, -F_i$ contain antipodal points such that $F_i \cap -F_i \neq \emptyset$. This sharply contrasts with the Borsuk-Ulam Theorem, which asserts there is a continuous map $f$ from $S^n$ into $\mathbb{R}^n$ over a surface containing antipodal surface vectors $\vec{p}, -\vec{p}$ such that $f(\vec{p}) = f(-\vec{p})$. Also, Theorem 1.4 concludes with the observation that the intersection of $F_i, -F_i$ is nonvoid.
but the values of the shared antipodal points are not given. In the LS theorem formulation, it is possible that the antipodal points in \( F_1 \cap -F_1 \) have different values. By contrast, in the Theorem 1.1 formulation, it is asserted that the antipodal points map to the same value.

Given a path \( h : I \rightarrow S^n \), let \( T = \{ t_i \} \) be an ordered and countable subset of \( I \), where \( 0 < t_i < t_{i+1} < 1 \) such that \( h(t_i) \neq h(t_{i+1}) \). We then have \( I_d = \{ 0, 1 \} \cup T \), which is called a discrete unit interval.

**Example 1.6.** Given a path \( h : I \rightarrow S^n \), let \( T_{0.0001} = \{ t_i \} \) be a countable and ordered subset of \( I \) such that \( 0 < t_i < t_j < 1 \) for all \( i < j \), and \( |h(t_i) - h(t_{i+1})| = 0.00001 \) for all \( i \). Then \( I_d = \{ 0, 1 \} \cup T_{0.00001} \) is a discrete unit interval.

## 2. Preliminaries

More recent versions of the Borsuk-Ulam Theorem (see, e.g., [11, §68,p.405], [19, p.266],[9, §2.1.p. 23]) require the map \( f : S^n \rightarrow \mathbb{R}^n \) to be continuous. The map \( f \) is continuous provided for each subset \( E \subset S^n \), if a point \( \vec{p} \) is arbitrarily close to \( E \) (i.e., \( \inf_{x \in E} |\vec{p} - \vec{x}| = 0 \)), then \( f(\vec{p}) \) is arbitrarily close to \( f(E) \). However, in keeping with an interest in the geometric realization of discrete paths as surface arcs containing points with gaps between them, we consider discrete maps.

**Definition 2.1.** Let \( S \) be a Riemannian surface. Given a path \( h : I \rightarrow S \), a discrete path is a map \( h : I_d \rightarrow S \) where \( I_d \) is a discrete unit interval of \( I \). (We will also denote the discrete path by \( h \).) Here \( \vec{h}(0) \) and \( \vec{h}(1) \) are the initial and terminal points in \( S \), respectively, and \( \vec{h}(t) \in S \) for all \( t \in I_d \).

**Example 2.2.** Discretely close surface points \( \vec{p}, \vec{q} \) such as close water molecules always have a minute gap between them.

**Example 2.3.** The discrete unit interval \( I_d \) is a collection of discretely close points \( t, t' \in I_d \) such that \( t' = t_{i+1} \).

**Definition 2.4.** A map \( f : S^n \rightarrow \mathbb{R}^n \) is said to be discrete provided for each subset \( E \subset S^n \), if a point \( \vec{p} \) is discretely close to \( E \), then \( f(\vec{p}) \) is close to \( f(E) \).

![Figure 2: The left-slanting arrow reads collapses to, e.g., \( \rightarrow \), i.e., collapse a left-pointing solid triangle to its boundary. For example, collapse a sphere \( S \) to a circle containing a discrete path \( h : I_d \rightarrow S \) with \( \vec{h}(0) = \vec{h}(x) \in \mathbb{R}^2 \), antipodal to \( \vec{h}(1) = \vec{h}(x) \in \mathbb{R}^2 \), with \( \vec{h}(t) \in \mathbb{R}^2 \) for \( t \in I_d \setminus [0, 1] \).](image)

**Example 2.5.** A sample discrete path \( h : I_d \rightarrow S \) on the surface of a Riemannian sphere is shown in Figure 2. This path begins at vector \( \vec{h}(0) \in \mathbb{R}^n \) at \( \vec{v}_1 \) on the surface of \( S \) and ends at vector \( \vec{h}(1) \in \mathbb{R}^n \), which is the value of antipode of \( \vec{v}_1 \). The assumption made here is that \( \vec{h}(0) \) and \( \vec{h}(1) \) have the same value such as identical temperature.

That is, a discrete path \( h : I_d \rightarrow S \) is a map from the discrete unit interval \( I_d \subset I \) (for \( I = [0, 1] \)) to a bounded, simply connected surface \( S \) with non-zero curvature. Path \( h \) is discrete, since there are gaps between all points \( \vec{h}(t) \in S \) between 0 and 1 in \( I_d \subset [0, 1] \). The surface \( S \) is simply connected provided every path \( h \) has
end points \( h(0), h(1) \in S \) and \( h \) has no self-loops.

Paths either lie entirely on a surface in the planar case or lie on a surface and, possibly, puncture a surface in the non-planar case. Paths that puncture a surface are called cross-cuts. A cross cut path \( P \) (also called an ideal arc [10, §3, p.11]) has both ends in \( P \) and path interior in the interior of \( S \).

**Remark 2.6.** Homotopic paths were introduced by J.H.C. Whitehead [21]. For Whitehead, a path \( h : [0, 1] \to X \) is a continuous map from the unit interval to a cell complex \( X \). In the pursuit of discrete paths in a curved space, the focus is on 0-cells (single points) and 1-cells (arcs) in an \( n \)-dimensional Riemannian space \( S \). A single surface vector is a 0-cell.

**Definition 2.7.** [5] An arc is a curvilinear line segment attached to a pair of 0-cells.

**Definition 2.8.** A pair of vectors \( v_0, v_1 \) is path-connected provided there is a sequence of 0-cells starting with \( v_0 \) and ending with \( v_1 \) in such a way that \( v_0, v_1 \) are attached to a Lefschetz arc. If such an arc exists between a pair of 0-cells \( v_0 \) and \( v_1 \) in this sequence (i.e., each pair \( v_0 \) and \( v_1 \) in the sequence of 0-cells are path connected), a collection of Lefschetz arcs corresponding to this sequence is called a discrete Lefschetz arc. We will denote the discrete Lefschetz arc between \( v_0 \) and \( v_1 \) by \( \overrightarrow{v_0, v_1} \).

**Proposition 2.9.** There is a discrete Lefschetz arc between each pair of 0-cells.

**Proof.** Immediate from Definition 2.8. \( \square \)

**Example 2.10.** All vectors on the circle in Figure 2 are path-connected, since, from Proposition 2.9, there is a Lefschetz arc between each pair of vectors.

3. Antipodal and Non-Antipodal Path Borsuk-Ulam Theorem

This section introduces results for the geometric realization of homotopic paths in surface arcs.

**Lemma 3.1.** Every discrete path constructs a discrete Lefschetz arc.

**Proof.** Given a path \( h : I \to X \), let \( h : I_d \to X \) be a discrete path. Then the collection

\[
\{h(0), h(1)\} \cup \{h(t_i) : t_i \in I_d\}
\]

forms a sequence of path connected 0-cells in \( X \), hence it forms a discrete Lefschetz arc between \( h(0) \) and \( h(1) \). \( \square \)

**Theorem 3.2.** The endpoints of a discrete Lefschetz arc can be the same.

**Proof.** Given two path connected 0-cells \( \overrightarrow{v_0} \) and \( \overrightarrow{v_1} \), we know that there is a discrete Lefschetz arc from \( \overrightarrow{v_0} \) to \( \overrightarrow{v_1} \). One can reverse the direction of arcs (since it can be considered to be a discrete path) so that the union of the discrete Lefschetz arcs \( \overrightarrow{v_0, v_1} \) and \( \overrightarrow{v_1, v_0} \) will form a discrete Lefschetz arc \( \overrightarrow{v_0, v_0} \). \( \square \)

Next, consider the geometric realization of discrete homotopic path as a discrete arc and which constructs a vector field.

**Theorem 3.3.** Every discrete path constructs a vector field.

**Proof.** Let \( h : I_d \to S \) be a discrete path. From Lemma 3.1, \( h \) constructs a discrete arc \( \overrightarrow{h(0), h(1)} \) on a surface \( S \). Consequently, each \( \overrightarrow{h(t)} \in h(0), h(1) \) has a location \((x_1, \ldots) \in S \) with its own magnitude and direction \( S \), i.e., every \( \overrightarrow{h(t)} \) is a vector in \( S \). Hence, \( h \) constructs a vector field. \( \square \)
**Lemma 3.4.** Let $\vec{v}_1, \vec{v}_2$ be antipodal vectors on the surface of an $n$-sphere $S^n$. There exists a discrete path $h$ with vectors that are antipodal on a surface $S^n$.

**Proof.** Let $\vec{v}_1, \vec{v}_2$ be antipodal vectors on the surface of an $n$-sphere $S^n$. Since $S^n$ is path connected, there is a discrete Lefschetz arc $\vec{v}_1, \vec{v}_2$. The collection of Lefschetz arcs (hence the discrete Lefschetz arc itself) forms a discrete path $h : I_d \to S^n$ with $\vec{h}(0) = \vec{v}_1$ and $\vec{h}(1) = \vec{v}_2$. Hence, a discrete path can be defined for every pair of antipodal points on $S^n$. \qed

From what we have observed about discrete paths on the surface of a sphere, we obtain the following theorem.

**Theorem 3.5.** (Path-Borsuk-Ulam Theorem) Given a continuous map $f : S^n \to \mathbb{R}^n$ (hence a discrete map), there exist a discrete path $h : I_d \to \mathbb{R}^n$ and a point $\vec{p} \in S^n$ such that $h(0) = f(\vec{p}) = f(-\vec{p})$. In fact, $h$ forms a discrete loop based at $f(\vec{p})$.

**Proof.** It is obvious that a continuous map $f : S^n \to \mathbb{R}^n$ is also a discrete map. From Theorem 1.1 (Borsuk-Ulam Theorem), we know that there is a point $\vec{p} \in S^n$ such that $f(\vec{p}) = f(-\vec{p})$. Consider a sequence of points $\{\vec{v}_t \in S^n\}$ indexed over a discrete interval $I_d$ such that $\vec{v}_0 = \vec{p}$, $\vec{v}_1 = -\vec{p}$, and two consecutive terms $\vec{v}_t, \vec{v}_{t+1}$ are discretely close for all $t \in I_d$. Then consider the image of this sequence $\{f(\vec{v}_t)\}_{t \in I_d}$. This set can be considered as the image of the discrete path $h : I_d \to \mathbb{R}^n$ defined by $h(t) = f(\vec{v}_t)$, $\vec{p}$ and two consequent terms $\vec{v}_t, \vec{v}_{t+1}$ are discretely close for all $t \in I_d$. Hence, $h$ forms a discrete loop. \qed

**Remark 3.6.** An immediate consequence of Theorem 3.5 is that, for any pair of antipodal surface points, we can always introduce a discrete path $h$ that begins and ends at the antipodal points such as places that have same latitude and longitude. For example, the antipode of Winnipeg, Manitoba, Canada with coordinates 49°.53’N, 97°.8’W is Port-aux-Français, Kerguelen, French Southern Territories.

**Example 3.7.** An example of a discrete path that begins and ends at antipodal surface points is shown in Figure 2.

Observe that a path can be constructed between any pair of surface vectors. This observation leads to more general form of Theorem 3.5.

**Theorem 3.8.** (Non-antipodal path-BUT) Let the discrete unit interval $I_d$ be an index set for vectors $v_0, \ldots, v_t, \ldots, v_1$, $t \in I_d$ in $S^n$ in a continuous map $f : S^n \to \mathbb{R}^n$ such that $f(v_0) = f(v_1)$ for some $v_0, v_1 \in S^n$. There is a discrete path $k : I_d \to \mathbb{R}^n$ with endpoints $f(v_0), f(v_1)$ that are values in $\mathbb{R}^n$ such that $k(0) = k(1)$.

**Proof.** Let $h : I \to S^n$ be a path from $v_0$ to $v_1$ and $h : I_d \to S^n$ be its associated discrete path. Then the composition $k = f \circ h$ is a discrete path in $\mathbb{R}^n$ with endpoints $f(v_0)$ and $f(v_1)$ so that $k(0) = k(1)$. \qed

![Figure 3: 2D and 3D views of discrete paths on a Gomboc Riemannian surface.](image)

**Example 3.9.** An example of a discrete path that begins and ends at antipodal surface vectors on a bumpy Riemannian sphere (aka Gomboc sphere) is shown in Figure 3.

**Example 3.10.** An example of a discrete path $h : S^2 \to \mathbb{R}^3$ on a 3D Gomboc Riemannian surface is shown in Figure 3. The same path is also depicted on a 2D slice of the 3D surface. In keeping with Theorem 3.8, each vector $\vec{h}(v_i)$ is a signal value from the path $h$. For example, if we let the discrete path be an optical field flow containing a stream of photons reflected from a Riemannian surface, then there are number of possible signal values for $\vec{h}(v_i)$, e.g.,
4. Feynman Trajectories of a Particle

This section introduces particle trajectories as continuous paths over the curvature of space-time, which leads to the counterpart of the discrete path results already given. The transition from discrete paths results from the geometry of space-time generated by quantum processes [8], which is in keeping with the observation by R. Penrose [13] that the link between space-time curvature and quantum processes such as those found in Feynman trajectory of a particle is supplied by the use of twistors. A twistor space is a complex manifold \( \mathbb{C}^M \). For example, a Lefschetz arc in curved space-time is a R.S. Ward hypersurface \( \mathbb{C}^S \).

**Example 4.1.** A sample twistor curve \( \ell \in \mathbb{C}^S \) is shown in Fig. 4, which is a geometric realization of a Feynman trajectory of a particle (see Def. 4.2), which leads to a space-time view of a Lefschetz arc (see Def. 4.2 and Lemma 4.5).

**Definition 4.2.** The trajectory of a particle in a 2-plane in curved space-time is a map

\[
h : \mathbb{R}^2 \times S^2 \to \mathbb{R}^2 \times S^2
\]

defined by

\[
h(t_i) = \overline{t_{i,0}t_i} \cup \bigcup_{t_i,t'_i} \overline{t_i,t'_i}, \quad t, t' \in \mathbb{R}, x_i \in S^2, i \in I,
\]

in which each \( \overline{t_{i,0}t_i} \) is a space-time line segment in a curve \( \ell \) starting with subarc \( \overline{t_{i,0}t_i} \) in a Lefschetz arc at time \( t_i \) (instant \( t \) in region time \( t \)) with index \( i \) in the unit interval \( I = [0,1] \) is mapped to an arcwise-connected set, i.e., the line segments in the trajectory are attached to each other and starting with \( \overline{t_{i,0}t_i} \), there is a path from any subarc a sequence of subarcs can be traversed to reach an ending subarc \( \overline{t_{i,n}t_i'} \) in a N.M.J. Woodhouse [23] twistor space \( \mathbb{R}^2 \times S^2 \) with metric signature \(+ + --\).
Remark 4.3. From Definition 4.2, the vectors in \( h(t_a) \) are J.H.C. Whitehead zero cells \([21]\) in an arcwise-connected space \( \mathbb{R}^2 \times S^2 \).

Definition 4.4. A Lefschetz arc \( E \) is a curve \( \ell \) attached between a pair of 0-cells \( p, p' \). We assume the curve \( \ell \) is dense and the points in \( \ell \) are path-connected, i.e., between every pair of points \( q, q' \) in \( \ell \), there is a sequence of sub-arcs traversable between \( q \) and \( q' \).

Lemma 4.5. A trajectory of a particle is realizable as a Lefschetz arc.

Proof. From Definition 4.2, a trajectory \( h \) is a curve \( \ell \) that starts and ends with a 0-cell and is the union of subarcs in an arcwise-connected space. Hence, from Definition 4.4, the trajectory \( h \) is realizable as a Lefschetz arc. \( \square \)

Example 4.6. A sample trajectory of a particle as a Lefschetz arc over a twistor curve realized as a Lefschetz arc \( \ell \) is the set of all real values in the closed interval with initial value 0 and ending 1 and an unbounded number of consecutive everywhere dense subintervals of real values between 0 and 1. That is, every real number \( x \) in a subinterval of \( A \subset I \) has another real number \( x' \) in \( A \) that is arbitrarily close to \( x \).

Definition 4.7. The unit \( I = [0, 1] \in \mathbb{R} \) is the set of all real values in the closed interval with initial value 0 and ending 1 and an unbounded number of consecutive everywhere dense subintervals of real values between 0 and 1.

Lemma 4.8. The trajectory of a particle is continuous.

Proof. From Definition 4.7, \( I \) is dense and is the index set for the points in the trajectory of a particle. A particle moving along the Lefschetz curve can be observed at any real value in the unit interval \( I = [0, 1] \) (see J.J. Sakurai and J. Napolitano [18, p. 37]). Let \( h \) be the trajectory of a particle \( t_a \). One can consider this trajectory as a curve \( \ell : I \rightarrow \text{Im } h \) defined by \( \ell(t) = t_{i_x} \) with \( \ell(0) = t_{i_a} \). Since \( \ell \) is continuous, for any close pair \( i, j \) in \( I \) will be mapped to close pair \( t_{i_x} \) and \( t_{t_x} \), and hence close points in \( \mathbb{R}^2 \times S^2 \) will be mapped to two close trajectories. Hence, \( h \) is continuous. Then if \( i, i' \in I \) are close, then \( t_{i_x}, t_{i_x'} \) are close. Hence, the trajectory \( h \) is continuous. \( \square \)

Remark 4.9. In the proof of Lemma 4.8, we considered a trajectory of a particle as a curve, parametrized on the closed interval \([0, 1] \). However, in 1-D Quantum Mechanics, this is not the case, i.e., the points of the trajectory may have an infinite number of possible values so that they may not be limited in \([0, 1] \) but rather are lying in \((-\infty, \infty) \). For more details, see J.J. Sakurai and J. Napolitano [18, pp. 37-42].

Example 4.10. Given a trajectory \( h \), consider the set \( J = \{ t_i \} \) of the instants of time of occurrence of the points in the trajectory of a particle over a vector field. The map \( g : I \rightarrow \mathbb{R} \) defined by \( g(t) = t_i \) is continuous, since for every arbitrarily close pair \( i \) and \( j, t_i \) and \( t_j \) are also arbitrarily close.

5. Feynman Path Integral

In this section, it is observed that a Feynman path is continuous (Lemma 5.2), which leads to the results in Theorem 5.4 and Theorem 5.5 for Feynman paths, which are consequences of the Borsuk-Ulam Theorem.

Definition 5.1. [4, p. 31] A Feynman path is a function \( H : \mathbb{R}^2 \times S^2 \rightarrow S^2 \) defined by \( H(t_{i_a}) = x_a \) for a particle at point \( x \) at time \( t_{i_a} \).

Lemma 5.2. Every Feynman path is continuous.

Proof. Let \( H : \mathbb{R}^2 \times S^2 \rightarrow S^2 \) be a Feynman path, defined by \( H(t_{i_a}) = x_a \) which is the trajectory \( h \) of a particle at point \( x_a \) at time \( t_{i_a} \). Let \( \ell \) represent that a particle travels over during its trajectory and let \( H(t_{i_{b}}) = x_b \) be a point in \( \ell \). For simplicity, the curve \( \ell \) is referred to as the trajectory of a particle. During the passage of a particle over \( \ell \), \( \ell \) has no gaps in it. Since a trajectory map \( h \) is continuous, two close points \( t_{i_{b}} \) and \( t_{i_{b'}} \) will lead us two close points \( x_b \) and \( x_{b'} \) in \( \ell \) at time \( t_{i_{b}} \). Hence, a Feynman path \( h \) is continuous. \( \square \)
Remark 5.3. In Lemma 5.2, the continuity of a Feynman path $H$ is explained in terms of the closeness (nearness) paradigm from [12, §1.5, p. 8], instead of the abstract (less intuitive) $\epsilon - \delta$ view of continuity. This approach befits the character of the trajectory of a particle over a curve $\ell$, where the trajectory of a particle and the curve $\ell$ (without gaps) are traced by the particle in its trajectory. Just as pairs of points in the curve $\ell$ can be arbitrarily close, so too, from Lemma 4.8, the vectors $H(t_{i_n}), H(t_{i_n})$ in the trajectory of a particle can be arbitrarily close.

The value of a path between points $a$ and $b$ on a curve $\ell$ (the positions of a particle trajectory at times $t_a, t_b$, respectively), is $K(b, a)$, defined in a complex space $CS$ with respect to Planck’s constant $\hbar$ by Feynman and Hibbs [4, p. 45] by

$$V(x, t) = \text{Potential energy of particle with mass } m.$$  
$$L = \frac{m}{2} x^2 - V(x, y) \text{ (Lagragian for the system).}$$  
$$S[b, a] = \int_{t_a}^{t_b} L(x, x, t)dt$$  
$$a, b = \text{points on a twistor curve.}$$  
$$K(b, a) = \int_{a}^{b} e^{(\frac{i}{\hbar})S[b, a]}\mathcal{L}x(t).$$

A Feynman path $H : \mathbb{R}^2 \times \mathbb{S}^2 \to \mathbb{S}^2$ over a curved space $\mathbb{S}^2$ can be considered as $H = pr_2 \circ h$, the composition of its corresponding trajectory map $h : \mathbb{R}^2 \times \mathbb{S}^2 \to \mathbb{R} \times \mathbb{S}^2$ and the second projection map $pr_2 : \mathbb{R}^2 \times \mathbb{S}^2 \to \mathbb{S}^2$. Given a fixed point $b_h$ on $\ell$, define $a : \mathbb{S}^2 \to \mathbb{R}^2$ by $a(\vec{\alpha}) = K(b_h, a)$ where $K(b_h, a)$ is the value of the trajectory $h$ containing points $b_h, a$ in a segment $b_h, a$ in a curve $\ell$ starting at $a$ and terminating at $b_h$.

Theorem 5.4. (Feynman Path Theorem) Given a map $\alpha : \mathbb{S}^2 \to \mathbb{R}^2$, there exists $\vec{a}$ in $\mathbb{S}^2$ such that $\alpha(\vec{a}) = \alpha(-\vec{a})$.

Proof. From Lemma 5.2, a Feynman path $H$ is continuous and so that $\alpha$ is also continuous. Hence, from Theorem 1.1, we obtain the desired result, $\alpha(\vec{a}) = \alpha(-\vec{a})$ for antipodal points $a, -a$ in a Feynman path $H$. \Box

Theorem 5.5. (Feynman Trajectory-of-Particle Theorem) The Feynman trajectory of a particle satisfies Borsuk-Ulam Theorem 1.1. Let $H : \mathbb{S}^2 \to \mathbb{R}^2$ be the trajectory of a particle on the surface of sphere. There is at least one pair vectors $\vec{p}, -\vec{p} \in \mathbb{S}^2$ such that $H(\vec{p}) = H(\vec{p})$.

Proof. From Lemma 5.2, a Feynman trajectory is continuous. Hence, from Theorem 1.1, we obtain the desired result for antipodal points $\vec{p}, -\vec{p} \in \mathbb{S}^2$ in the Feynman trajectory $h$. \Box

Theorem 5.6. (Feynman Path Integral Theorem) There exists a Feynman path with an initial path integral $K(b_h, a)$ for an initial vector $\vec{a}$ that equals the path integral $K(b_h, -a)$ for a later vector $-\vec{a}$, which may or may not be the antipode of vector $\vec{a}$.

Proof. $K(b_h, a)$ are Feynman path integrals that resonate (have values) for a particle that has gradients on any two different surface curvatures along a surface curve $\ell$ or on the same surface curvature on a path $\ell'$ for a boomerang trajectory that follows a path that is a cycle. In either case, choose an intermediate point $b_h$ in the path between $\vec{a}$ and $b_h$ so that the two segments on $\ell$ have the same length. In that case, $K(b_h, a) = K(b_h, -a)$. \Box

Remark 5.7. The significance of Theorem 5.6 is that the endpoints on a particle trajectory curve $\ell$ need not be antipodal points. That is, Theorem 5.6 is more general than Theorem 5.4.
6. Woodhouse Borsuk-Ulam Theorem

This section gives three results for N.M.J. Woodhouse contour integrals [23, p. 198], defined with respect to the set of all real \( \alpha \)-planes that has topology \( \mathbb{R}^2 \times S^1 \), which is compactified by adding \( S^1 \) representing \( \alpha \)-planes that lie in the null cone at \( \infty \). First, consider

\[ \xi = x_1 + ix_2 \] and \( \tau = t_1 + it_2 \), representing \( \alpha \)-planes as surfaces, with

\[ w = \xi + z\tau, \quad \bar{w} = \bar{\xi} + z\bar{\tau}, \] constant for \( z = e^{i\theta} \), where

\[ z = e^{i\theta} \] determines orientation of \( \alpha \)-plane.

\[ \phi(w, \bar{w}, z) = \frac{1}{2\pi} \oint_{|z|=1} f(w, \bar{w}, z) \frac{dz}{z}, \] expanded to obtain

\[ \phi(w, \bar{w}, z) = \frac{1}{2\pi} \oint_{|z|=1} f(\xi + z\tau, \bar{\xi} + z\bar{\tau}, z) \frac{dz}{z}. \]

Let \( \Phi : S^2 \to \mathbb{C} \) be a map defined by \( \Phi(p) = \phi(w_p, \bar{w}_p, z) \), where \( w_p \) is the point representing \( p \) on the equilateral circle \( S_2 \) on \( S^2 \) which is passing through \( p \). The function \( \Phi \) can be realized as a function \( \Phi : S^2 \to \mathbb{R}^2 \) as \( \mathbb{C} \) and \( \mathbb{R}^2 \) are homeomorphic.

**Definition 6.1.** The contour integral \( \Phi : S^2 \to \mathbb{R}^2 \) is a smooth function, since \( \phi \) is a smooth function on a twistor space [23]. That is, \( \Phi \) is continuous.

![Figure 5: Woodhouse contour integrals on sub-twistor curve antipodes](image)

**Theorem 6.2.** The contour integral \( \Phi \) satisfies the Borsuk-Ulam Theorem.

**Proof.** From Definition 6.1, the contour integral \( \Phi : S^2 \to \mathbb{R}^2 \) is a continuous function. The result follows from Theorem 1.1, i.e., there exist antipodes \( \bar{p}, -p \) on a twistor curve in \( \mathbb{R}^2 \times S^2 \) such that \( \Phi(p) = \Phi(-p) \).

**Corollary 6.3.** The map \( \Phi \) also satisfies the path-Borsuk-Ulam Theorem given in Theorem 3.5.

**Proof.** Take \( n = 2 \) and replace the continuous map \( f : S^n \to \mathbb{R}^n \) with \( \Phi : S^2 \to \mathbb{R}^2 \) in the proof of Theorem 3.5.

**Example 6.4.** Sample contour integrals on sub-twistor vectors that are antipodal are shown in Figure 5.

**Theorem 6.5.** Let \( \phi, \phi' \) be the Woodhouse contour integrals over a twistor curve \( \ell \) and let \( p, p' \) be any two distinct points on \( \ell \). Then there are \( \Phi, \Phi' \) such that \( \Phi(p) = \Phi(p') \).

**Proof.** Replace the Feynman path integral with the Woodhouse contour integral in the proof of Theorem 5.6, and the desired result follows. That is, we can always find a point \( q \) between \( p, p' \) on the twistor \( \ell \) such that \( \Phi(p) = \Phi(p') \).

**Remark 6.6.** Theorem 6.5 covers a broader spectrum of twistor length measurements than Theorem 6.2. That is, for any pair of distinct vectors on a twistor curve, we can always find an intermediate vector so that the contour integrals over the resulting twistor sub-arcs have equal value.

**Example 6.7.** Sample contour integrals on sub-twistor curves \( \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5 \) with end points that may or may not be non-antipodal are shown in Figure 6.
7. Concluding Remarks

The focus in path Borsuk-Ulam Theorem 3.5 is on a homotopic path between antipodes on the surface of a sphere $S^n$ mapped to real values in $\mathbb{R}^n$. The geometry underlying the Borsuk-Ulam Theorem looms up, for example, in the realization of a homotopic path as an arc stretching over a planetary curved surface between one location and another location at varying space-times with the same latitude and longitude. In this paper, the Borsuk-Ulam Theorem is an emperor with new clothes, namely,

1° **How to look:** consider either a discrete or continuous homotopic paths between antipodes.
2° **Geometric realization:** endpoints of twistor curves that are either antipodal or non-antipodal.
3° **Length-of-arc measure:** e.g., measure with either a Feynman path integral or Woodhouse contour integral over arcs having antipodal endpoints.

References